Nonlinear Oscillations in Chemical Networks as Symmetries of Transformation Groups

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The description of phenomena created by nonlinear oscillators with the aid of transformation groups has been attempted. Such a treatment of the chemical model of Dreitlein and Smoes was met with success. The mathematical approach is based on an application of the canonical form theorem for one-parameter groups. Conditions for the occurence of limit cycles have been derived.

1. Introduction

Nonlinear dynamic systems play an important role in all domains of nature. Their mathematical description as opposed to that of linear systems, can be very complicated, or even be an impossible task. The physical phenomena governed by nonlinear processes are far more diversified and rewarding objects for a study than those which obey linear laws. A typical example of a nonlinear phenomenon is the occurrence of a limit cycle. The latter manifests itself as a stable, sustained oscillation in a physical system. Such oscillations may often be identified in real systems and have therefore been investigated in several different fields of science [1–7]. The emphasis in such projects has mainly been placed on the identification of a limit cycle as the source of the oscillation observed and on the stability of the limit cycle. Here, these questions will not be examined.

The aim of the paper is to clarify the theoretical connection between dynamic systems, one-parameter transformation groups and limit cycles. The theory of the description of one-parameter transformation groups with differential equations has adequately been presented by other authors [8–11]. Here, we shall consider the system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ where x and y are time dependent variables which characterize the state of the system. $P = P(x, y)$ and $Q = Q(x, y)$ are analytic functions of x and y and are not explicitly time dependent, i. e. autonomous, $\dot{x}$ and $\dot{y}$ are the time derivatives of x and y resp.

At first, the concept is applied to the nonlinear kinetic model of Dreitlein and Smoes [6]. This model illustrates the relationship between a one-parameter group and the limit cycle of a dynamic system. The conditions for the induction of a limit cycle by a one-parameter group are then defined using the canonical form theorem. They impose restrictions to the functions P and Q and facilitate the construction of model systems. Finally, the statement derived about limit cycles will be recasted into a different mathematical form and extended to systems with more than two state variables. The implications of these principles to biological systems have been discussed elsewhere [12] and are therefore only mentioned in this paper.

2. A Chemical Model System

Dreitlein and Smoes [6, 13] have studied the kinetics of an oscillatory system consisting of two species and governed by the reaction equations:

$$\dot{c}_1 = k_1 (c_2 - a_2) + (c_1 - a_1) G_1 ,$$
$$\dot{c}_2 = -k_2 (c_1 - a_1) + (c_2 - a_2) G_2 \qquad (1)$$

where $c_i = c_i(t)$ denotes the concentration (or density) of the molecular (or biological) species. The quantities k_1, k_2, a_1 and a_2 are positive and constant parameters while:

$$G_i = E_i - \frac{1}{k} \cdot k_2 \cdot (c_1 - a_1)^2 / c_*^2 - \frac{1}{k} \cdot k_1 \cdot (c_2 - a_2)^2 / c_*^2$$

where $i = 1, 2$ and E_1, E_2 and c_* are additional, constant parameters. When the time units are so chosen that

$$k = \sqrt{k_1 \cdot k_2} \quad \text{and:}$$
$$x = (k_2/k_1)^{1/4} (c_1 - a_1)/c_* ,$$
$$y = (k_1/k_2)^{1/4} (c_2 - a_2)/c_* ;$$

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Eq. (1) will be dimensionless:

$$\dot{x} = (E - x^2 - y^2)x + k\cdot y - 2\Delta\cdot x\,,$$
$$\dot{y} = (E - x^2 - y^2)y - k\cdot x + 2\Delta\cdot y \qquad (2)$$

with $E = (E_1 + E_2)/2$ and $\Delta = (E_2 - E_1)/4$.

Dreitlein and Smoes[6] have proposed a simplified chemical reaction scheme with $k = 2$ for the system (1). At first, the simple case of $\Delta = 0$, i. e. $E = E_1 = E_2$, is examined. The Eqs. (2) then have only one singular point $(x = y = 0)$. In polar coordinates, i. e. $r^2 = x^2 + y^2$ and $\operatorname{tg}\varphi = y/x$, they are given by:

$$\dot{r} = r(E - r^2)\,, \quad \dot{\varphi} = -k\,. \qquad (3)$$

The solution of the equation system is:

$$r = \left(\frac{E}{1 + (E/r_0{}^2 - 1)\,e^{-2Et}}\right)^{1/2}, \qquad (4)$$
$$\varphi = -k\cdot t + \varphi_0\,.$$

At $t = 0$ are $r = r_0$ and $\varphi = \varphi_0$. A phase plane portrait of Eq. (4) is shown in Figure 1. For $E > 0$, the equation of the limit cycle is $r^2 = E$ and if $E < 0$, no limit cycle exists.

It may be verified that Eq. (4) forms a one-parameter (t) group of transformations. It transforms (r_0, φ_0) into (r, φ).

$$A = r(E - r^2)\frac{\partial}{\partial r} - k\frac{\partial}{\partial\varphi} \qquad (5)$$

is its infinitesimal generator. The finite transformations may subsequently be obtained by the operations:

$$r = e^{A\cdot t}\,r)_0 \text{ and } \varphi = e^{A\cdot t}\,\varphi)_0 \qquad (6)$$

$)_0$ = after the operations insert $r = r_0$ and $\varphi = \varphi_0$.

The system (3) remains invariant in form by application of the following groups:

The one-parameter (α) group of translations

$$\varphi^* = \varphi + \alpha\,; \quad r^* = r \qquad (7)$$

and a group of the order two:

$$\varphi^* = -\varphi\,; \quad t^* = -t\,; \quad (r^*)^2 = E - r^2\,. \qquad (8)$$

The transformation (8) uniquely maps the internal trajectories on those outside the limit cycle (see Figure 1). The infinitesimal generator (5) carries a point along its trajectory and generates the latter entirely for $-\infty < t < \infty$. The group (7) is of particular interest. Its generator is:

$$U = \partial/\partial\varphi\,. \qquad (9)$$

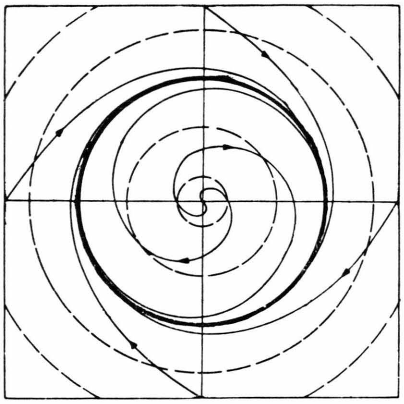

Fig. 1. Trajectories (solid line) and limit cycle (heavy solid line) for system (3). The dashed lines are the integral curves of the infinitesimal generator U of the one-parameter group (27).

This group rotates the trajectories of Fig. 1 by the angle α around the origin. It transforms, because of the rotational symmetry, trajectories into conformal trajectories. Hence, the knowledge of one trajectory permits the generation of new trajectories of system (3) using the transformations (7). Thus, the groups of transformations generate all trajectories of the system (3) and hence its solution.

Next, we consider the notion of invariance under a group. For all values, Eq. (4) maps points of a trajectory into other points of it. A trajectory is an invariant curve since it is transformed into itself by every transformation (4). All invariant curves $f(x, y)$ may be determined from the infinitesimal generator (5) as solutions of the partial differential equation:

$$A\,f = 0 \text{ whenever } f = 0\,. \qquad (10)$$

The transformation of trajectories into one another by the group (7) is another example of an invariance under a group. The transformation of the trajectory $f(r, \varphi)$ into $f^*(r, \varphi)$ by the group (9) may be written as:

$$f^* = U\,f$$

where f^* is a function of f.

The condition that U transforms the trajectories $f(r, \varphi)$ of A into one another may be expressed by the commutator of U and A:

$$[U, A] = \lambda\cdot A \qquad (11)$$

where λ is a function of the variables r and φ.

These relations may be verified by recasting (4) into:

$$f = (E/r^2 - 1) \exp\{-2(E/k)\varphi\} . \qquad (12)$$

An examination of Fig. 1 reveals that the group (7) yields new trajectories. The limit cycle is an exception since it, like the singular point at the origin, is an invariant curve of U and therefore always is transformed into itself.

All trajectories touch these invariants but do not intersect them. Otherwise, the unique axiom of the group would be invalidated. From a dynamic point of view, these invariants are of special interest since they, as limit trajectories, represent states in which the system, in the absence of external perturbations, will remain indefinitely. The limit cycle of nonlinear oscillations in a chemical system and the conditions for its arisal will in the following be given special consideration since this phenomenon is of great importance, e. g. to biology.

The limit trajectories may often be determined:

1. by the extrapolation $t \to \infty$,
2. by a search for trajectories which are common to the groups defined by U and A or
3. by location of envelopes to the trajectories.

The trajectories loose their uniqueness when they become limiting. They then acquire the symmetry properties of both of the infinitesimal transformations of U and A. The latter function generates the periodicity in the parameter t which is observed as an oscillation in recorded quantities.

Now, we briefly consider the case where $E_1 \neq E_2$. This case may, as will be shown in the following chapters, be treated by use of the canonical form theorem and the transformations:

$$\operatorname{tg}(u) = [a_1(y/x) + a_2]/[a_3(y/x) + 1]$$
$$v^2 = a_4 x^2 + a_5 x \cdot y + a_6 y^2$$

where a_1 through a_6 are constants.

The limit cycle only exists for certain ranges of E_1 and E_2 [6]. If the values of E_1 and E_2 are so chosen that a limit cycle exists, the latter may be represented by a curve of second order:

$$f(x, y) = a_4 x^2 + a_5 x y + a_6 y^2 + a = 0 . \qquad (13)$$

The quantities a_4, a_5, a_6 and a (where $a \neq 0$) depend on E_1, E_2 and k. The function (13) may define an ellipse.

3. One-Parameter Group of Translations

We now consider in more details the one-parameter (α) group of translations parallel to the u-axis:

$$u^* = u + \alpha , \quad v^* = v . \qquad (14)$$

The notation u and v is here chosen for the coordinates to save x and y for the Cartesian system. The infinitesimal generator of the group is

$$U = \partial/\partial u . \qquad (15)$$

The extended generator U' is identical to U and therefore the group is also identical to the extended one. The general differential equation in the variables u and v which is invariant under the group is $dv/du = F(v)$, where F can be any analytic function. The definition of invariance of a differential equation under a transformation group is given in the literature [8–11].

If $\quad F(v) = f(v) \cdot h(u, v) / (g(v) \cdot h(u, v))$ and $g(v) \neq 0; \quad h(u, v) \neq 0,$

then the corresponding most general system of differential equations is:

$$\dot{u} = g(v) \cdot h(u, v) ,$$
$$\dot{v} = f(v) \cdot h(u, v) . \qquad (16)$$

This system has the infinitesimal generator

$$A = g(v) \cdot h(u, v) \frac{\partial}{\partial u} + f(v) \cdot h(u, v) \frac{\partial}{\partial v} . \qquad (17)$$

The criterion that a differential equation (defined by its infinitesimal generator A) admits a group (defined by its infinitesimal generator U) may be expressed in the form of Eq. (11) with:

$$\lambda = \frac{1}{h} \cdot \frac{\partial h}{\partial u} . \qquad (18)$$

So far, the development of the theory has been independent on whether or not the dynamic system contains a limit cycle. The latter may be defined as a closed trajectory which is isolated, i. e. does not belong to a continuous family of closed trajectories [14].

A necessary condition for limit cycles may be found by examination of the family of integral curves of the differential equations (16). If $\Phi(u, v) = 0$ is such an integral curve and if $U\,\Phi = 0$ identically or through $\Phi(u, v) = 0$, then the function Φ is said to be also invariant with respect to the translation group (14). Such a curve does not need exist.

Otherwise, we have:

$$U\Phi = 0 \text{ whenever } \Phi = 0\,, \quad A\Phi = 0\,. \tag{19}$$

The common solution $\Phi(u,v)$ to U and A must satisfy the conditions (19), i. e.:

$$\partial\Phi/\partial u = 0 \text{ from } U\Phi = 0\,,$$

$$g(v)\cdot h(u,v)\,\partial\Phi/\partial u + f(v)\cdot h(u,v)\,\partial\Phi/\partial v = 0 \text{ from } A\Phi = 0\,. \tag{20}$$

The existence of a non-trivial common solution $\Phi(u,v)$ requires that the determinant is zero:

$$h(u,v)\cdot f(v) = 0 \tag{21}$$

which restricts Φ. Thus, the necessary condition for the existence of limit cycles (21), neither depends on $g(v)$, nor on the initial state of the system.

4. The Canonical Form Theorem and Coordinate Transformations

The relations described above will now be extended to other coordinates. The new (also non-orthogonal) coordinates (u^*, v^*) of any point will be expressed as functions of u and v:

$$u^* = u^*(u,v)\,, \quad v^* = v^*(u,v)\,. \tag{22}$$

We assume that Eq. (22) may be solved for u and v as functions of u^* and v^*:

$$u = u(u^*, v^*)\,; \quad v = v(u^*, v^*)\,. \tag{23}$$

The functions (22) and (23) are assumed to be single valued and to possess continuous derivatives, i. e. that the correspondence between (u^*, v^*) and (u, v) is unique. This assumption may not be valid at certain points. They must therefore be considered separately.

Under this coordinate transformation, infinitesimal generators U of a group are converted to their new form U^*, according to the rule [8]:

$$U^* = (U\,u^*)\,\partial/\partial u^* + (U\,v^*)\,\partial/\partial v^*\,. \tag{24}$$

The generator of the extended group of point transformations U', (24), is then:

$$U'^* = (U'\,u^*)\,\partial/\partial u^* + (U'\,v^*)\,\partial/\partial v^* + (U'\,p^*)\,\partial/\partial p^*$$

where

$$p^* = \frac{\mathrm{d}v^*}{\mathrm{d}u^*} = \frac{\partial v^*/\partial u + p\,\partial v^*/\partial v}{\partial u^*/\partial u + p\,\partial u^*/\partial u}$$

and $\quad p = \mathrm{d}v/\mathrm{d}u\,.$

The system (16) then acquires the form:

$$\dot u^* = g(v)\cdot h(u,v)\,\frac{\partial u^*}{\partial u} + f(v)\cdot h(u,v)\,\frac{\partial u^*}{\partial v}\,,$$

$$\dot v^* = g(v)\cdot h(u,v)\,\frac{\partial v^*}{\partial u} + f(v)\cdot h(u,v)\,\frac{\partial v^*}{\partial v} \tag{25}$$

where $u = u(u^*, v^*)$ and $v = v(u^*, v^*)\,.$

The chemical model will now be transformed:

$$v = r\,, \quad u = \varphi \text{ and } u^* = x = r\cos\varphi\,,$$
$$v^* = y = r\sin\varphi\,,$$

i. e. u and v may be regarded as polar coordinates and x and y as Cartesian.

The group (14) then takes the form:

$$x^* = x\cos\alpha - y\sin\alpha\,,$$

$$y^* = x\sin\alpha + y\cos\alpha\,. \tag{26}$$

The parameter α is the angle of rotation.

According to Eq. (24), the infinitesimal generator (15) induced by the group (26) is:

$$U_{xy} = -y\,\partial/\partial x + x\,\partial/\partial y\,. \tag{27}$$

The extended group U'_{xy} is now no longer identical to (27), but may be expressed as:

$$U'_{xy} = -y\,\partial/\partial x + x\,\partial/\partial y + (1 + y'^2)\,\partial/\partial y' \tag{28}$$

where $y' = \mathrm{d}y/\mathrm{d}x$.

Finally, the most general differential equation $H(x, y, y') = 0$ which satisfies the group (28) may be obtained from its invariance $U'_{xy}\,H = 0$, whenever $H = 0$, as $H = F(v) - p$. The associated differential system (16) is:

$$\dot x = \frac{x}{r}\,f(r)\,h^*(x,y) - y\,g(r)\,h^*(x,y)\,,$$

$$\dot y = \frac{y}{r}\,f(r)\,h^*(x,y) + x\,g(r)\,h^*(x,y)\,. \tag{29}$$

The condition which must be fulfilled by a common solution to the infinitesimal generator of the groups (20) has the form:

$$\left\{\frac{x}{r}\cdot f(r)\cdot h^*(x,y) - y\,g(r)\,h^*(x,y)\right\}\frac{\partial}{\partial x} + \left\{\frac{y}{r}\,f(r)\,h^*(x,y) + x\,g(r)\,h^*(x,y)\right\}\frac{\partial}{\partial y} = 0$$

$$-y\,\frac{\partial}{\partial x} + x\,\frac{\partial}{\partial y} = 0\,. \tag{30}$$

Equation (21) becomes in general

$$h(u,v)\cdot f(v)\cdot\frac{\partial(u^*,v^*)}{\partial(u,v)}=0$$

and in the present case:

$$r\cdot f(r)=0 \tag{31}$$

which on application to the simplified chemical model system by comparison with (29) yields:

$$f(r)=(E-r^2)\,r \quad\text{and}\quad g(r)=-k\,.$$

Hence, $r=0$ and $f(r)=0$ represent a singular point and a limit cycle of the system, respectively.

The condition (30) may also be obtained from (19) by the rule (24). In that case it has the form:

$$\begin{aligned}&(A\,u^*)\frac{\partial}{\partial u^*}\Phi+(A\,v^*)\frac{\partial}{\partial v^*}\Phi=0\\ &\qquad\qquad\text{whenever }\Phi=0\,.\\ &(U\,u^*)\frac{\partial}{\partial u^*}\Phi+(U\,v^*)\frac{\partial}{\partial v^*}\Phi=0\end{aligned} \tag{32}$$

This expression may, by introduction of the functions P, Q, ξ and η and by setting $u^*=x$ as well as $v^*=y$, be converted to:

$$P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}=0\,,\qquad \xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}=0\,. \tag{33}$$

The form of (21) and (31) may be derived from (33) as

$$\xi\,Q-P\,\eta=0\,. \tag{34}$$

The associated system of differential equations (25) is then:

$$\dot x=P(x,y)\,,\quad \dot y=Q(x,y)\,. \tag{35}$$

So far, ξ and η have been derived by a coordinate transformation, now these functions are considered as given and a transformation is sought which carries ξ and η to their canonical form, $\xi=0$ and $\eta=1$. The canonical form theorem [8, 11] states that such a transformation exists. Furthermore, to any differential equation belonging to (33) a ξ, η may be found [8–11]. It should therefore be possible to retrace the derivation in backward direction from (35) to the translational form of the group. This procedure, however, is not always feasible. It may be mentioned that $(\xi\,Q-\eta\,P)$ [compare (34)] is an integrating factor of the differential equation $Q\,dx-P\,dy=0$ which describes the trajectories of the system (35), and that the construction of models may be facilitated by the above coordinate transformation.

5. Extension of the Criteria

Also systems of higher order than two exhibit limit cycles [15]. An extension of the theory to three or more dimensions is therefore required. The one-parameter (α) translation group and its infinitesimal generator are then:

$$u_1=u+\alpha\,;\quad v_1=v\,;\quad w_1=w \quad\text{and}\quad U=\partial/\partial u\,. \tag{36}$$

The general operator A, satisfying Eq. (11) for U, is:

$$\begin{aligned}A=&\,g(v,w)\cdot k(u,v,w)\,\partial/\partial u\\ &+f(v,w)\cdot k(u,v,w)\,\partial/\partial v\\ &+h(v,w)\cdot k(u,v,w)\,\partial/\partial w\,.\end{aligned} \tag{37}$$

This operator is associated with the system:

$$\begin{aligned}\dot u&=g(v,w)\cdot k(u,v,w)\,,\\ \dot v&=f(v,w)\cdot k(u,v,w)\,,\\ \dot w&=h(v,w)\cdot k(u,v,w)\,.\end{aligned} \tag{38}$$

The condition, which must be fulfilled by a common solution such as (19), is:

$$\left.\begin{aligned}A\,\Phi&=0\\ U\,\Phi&=0\end{aligned}\right\}\ \text{whenever }\ \Phi=0\,. \tag{39}$$

Equations (39) form a complete system. Their solution $\Phi=0$ is, in general, a surface in a threedimensional space. A common trajectory may exist as the intersection of such surfaces. Moreover, from the condition of a common tangent vector to the group of U and A, one obtains $f(v,w)=0$ and $h(v,w)=0$.

The operator A may, in addition to (36), be invariant under a second one-parameter (β) group, e. g.:

$$u_1=u\,;\quad v_1=v+\beta\,;\quad w_1=w \tag{40}$$

with the infinitesimal generator $U_\beta=\partial/\partial v$. In that case, the general form of $\tilde A$ is restricted to:

$$\begin{aligned}\tilde A=&\,\tilde g(w)\cdot k(u,v,w)\frac{\partial}{\partial u}+\tilde f(w)\cdot k(u,v,w)\frac{\partial}{\partial v}\\ &+\tilde h(w)\cdot k(u,v,w)\frac{\partial}{\partial w}\end{aligned} \tag{41}$$

with the associated differential equations:

$$\begin{aligned}\dot u&=\tilde g(w)\cdot k(u,v,w)\,,\\ \dot v&=\tilde f(w)\cdot k(u,v,w)\,,\\ \dot w&=\tilde h(w)\cdot k(u,v,w)\end{aligned} \tag{42}$$

where $k(u,v,w)\neq 0$.

Now, the condition for the existence of a common solution (19) is:

$$\tilde{A}\,\Phi = 0\,;\quad U_\alpha\,\Phi = 0\,;\quad U_\beta\,\Phi = 0 \quad \text{whenever} \quad \Phi = 0\,. \tag{43}$$

The determinant of the system must be zero to permit a nontrivial solution:

$$h(w)\,k(u,v,w) = 0\,. \tag{44}$$

This equation is equivalent to Equation (21).

The procedure may be extended to canonical forms of more-parameter groups as well as to more dimensions than three.

Finally, the mutual relations between point and contact transformations will be treated. A contact transformation is a transformation of lineal elements (x, y, p) into their like, (x^*, y^*, p^*):

$$x^* = x^*(x,y,p)\,;\quad y^* = y^*(x,y,p)\,;\quad p^* = p^*(x,y,p) \tag{45}$$

under preservation of the differential form:

$$\mathrm{d}y^* - p^*\,\mathrm{d}x^* = \varrho(x,y,p)\cdot(\mathrm{d}y - p\,\mathrm{d}x) \tag{46}$$

where p and p^* are derivatives of y with respect to x and y^* with respect to x^*, respectively. The infinitesimal generator B of a one-parameter group of contact transformations is[8]:

$$B = \frac{\partial W}{\partial p}\frac{\partial}{\partial x} + \left(p\,\frac{\partial W}{\partial p} - W\right)\frac{\partial}{\partial y} - \left(\frac{\partial W}{\partial x} + p\,\frac{\partial W}{\partial y}\right)\frac{\partial}{\partial p} \tag{47}$$

where $W(x,y,p) = p\,\xi - \eta$ is its characteristic function. If $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$ are independent of p, then B is the infinitesimal extended group transformation to the group of (33):

$$U = \xi\,\partial/\partial x + \eta\,\partial/\partial y\,. \tag{48}$$

Hence, the condition to be fulfilled, if the differential equation $H(x,y,\mathrm{p}) = 0$ is to be invariant under U, may also be expressed as:

$$B\,H = 0 \quad \text{whenever} \quad H = 0\,. \tag{49}$$

A special case arises if B is applied to W:

$$B\,W = -\frac{\partial W}{\partial y}\cdot W\,. \tag{50}$$

The equation $B\,W = 0$ may then be satisfied in two different ways:

1. $\partial W/\partial y = 0$, which yields a canonical contact transformation with $\varrho = 1$. This case is well known from mechanics.
2. $W = 0$, which is equivalent to Eq. (49) with $W(x,y,p)$ as a differential equation. If $H = P\cdot p - Q = 0$ is chosen as the differential equation, then the infinitesimal generator U, which may be obtained from B for $p\,\xi - \eta = 0$, only selects those trajectories which have a lineal element in common with H. Consequently:

$$p = \eta/\xi = Q/P$$

which is Equation (34). This condition identifies the limit cycle as an envelope of trajectories.

In the simplified chemical model the characteristic function is $W = -x - y\,p$. With $\xi = -y$ and $\eta = x$ one obtains the same result as above.

6. The Significance of Limit Cycles and Singular Points to Biological and Chemical Systems

In the preceding sections, a mathematical procedure to derive the limit cycles and the critical points of a dynamic system has been described. Basicly, the procedure consists of submitting the system to the special condition that the solution must be symmetrical under a given one-parameter group. This requirement is only satisfied by a few solutions of the system, viz. those representing the invariances.

The physical implications of invariances are illustrated by the chemical model system. Figure 1 shows that the system approaches the limit cycle and the singular point along trajectories. Each of these is specified by the initial conditions of the system. If the system moves on the limit cycle, its approach path can no longer be specified. This multiplicity is a general property of limit cycles. It implies that the initial conditions of the system from this stage of the development and onwards have lost their significance. The thermodynamic equilibrium of a chemical reaction system is an example of such a state. It is therefore impossible to identify the path of approach once the equilibrium has been attained. Since a closed thermodynamic system invariably moves toward its equilibrium state it can neither display limit cycles nor multiple steady states. Conversely, the existence of a symmetric solution of a system under a one-parameter group implies that the system is open. Limit cycles and multiple singular points are also inconsistent with the Onsager relation since the latter only is valid near the thermodynamic equilibrium. If therefore the condition of symmetry under a parameter group is satisfied, the system must remain far from equilibrium. Both arguments may be verified with the chemical model system. The former is fulfilled by maintaining the concentrations of one or more reactants constant and the second by suppressing the reverse direction of some of the reactions. All biological systems operate under conditions where both of these features exist.

A state of a system may be stable, semistable or unstable, but only a stable state may be maintained for an extended period. The analysis of the stability of states has, however, been adequately treated elsewhere and will therefore not be discussed here. However, some points of particular interest should be outlined. The stability of a state is important when several interconvertible states exist. The former always depends on the parameters of the system. In the chemical model system, the quantity E is such a parameter. If E is changing signs, an abrupt change in the state of the system is precipitated. Limit cycles do not exist if $E>0$ while $E=0$ induces a phase transition (or catastrophe). A phase diagram is required to describe the properties of the system if more than one parameter can vary (compare[6]).

The phase consideration brings about another aspect. By the symmetry, along with the initial conditions, many of the details of the system have been eliminated. One therefore views the system from a different level, where only the singular points and the limit cycles determine the dynamics. The transition from the quantum mechanical picture of a process to its mechanical equivalent resembles this view. Large systems may contain more than one element of symmetry and therefore may display a multiplicity of phenomena, e. g. regulatory and rhythmic processes. Also the introduction of the space coordinate should result in a new class phenomena.

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